Topology Vol. 14, pp. 173–177. Pergamon Press. 1975. Printed in Great Britain

ON A THEOREM OF PITTIE

ROBERT STEINBERG

(*Received* 1 *October* 1974)

§1. INTRODUCTION

HARSH V. PITTIE[3] has proved the following result:

THEOREM 1.1. *Let G be a connected compact Lie group with $\pi_1 G$ free and G' a (closed) connected subgroup of maximal rank. Then $R(G')$ is free (as a module) over $R(G)$ (by restriction).*

Here $R(G)$ denotes the complex representation ring of G. For the bearing of (1.1) on the K-theory of G the reader may consult [3]. Pittie's proof actually omits a few cases, which can however be checked out by hand. Here we present an elementary proof which yields an explicit basis for $R(G')$ over $R(G)$ (see (2.2) and (2.3(a)) below) and then a converse after suitably weakening the assumption on $\pi_1 G$.

THEOREM 1.2. *Let G be a connected compact Lie group and S its semisimple component. Then the following conditions are equivalent.*

(a) *$R(G')$ is free over $R(G)$ for every connected subgroup G' of maximal rank.*

(b) *$R(T)$ is free over $R(G)$ for some maximal torus T.*

(c) *$R(G)$ is the tensor product of a polynomial algebra and a Laurent algebra.*

(d) *$R(S)$ is a polynomial algebra.*

(e) *S is a direct product of simple groups, each simply connected or of type SO_{2r+1}.*

Since $\pi_1 G$ is free if and only if S is simply connected, because G is the product of S and a central torus, the equivalence of (a) and (e) provides the just-mentioned extension and converse of (1.1).

As a result of our development we also obtain:

THEOREM 1.3. *Theorems 1.1 and 1.2 are true for linear algebraic groups over algebraically closed fields (instead of compact Lie groups) and their rational representations.*

§2. PROOF OF (1.1)

We may, and shall, assume that G is semisimple, hence simply connected since $\pi_1 G$ is free, as is indicated in [3]. Let T be a maximal torus of G', hence also of G, and W' and W the corresponding Weyl groups, and X the character group (lattice) of T. As is known (see [1]), $R(G)$ may be identified with $\mathbf{Z}[X]^W$ via restriction to T, even if G is not semisimple, and similarly for $R(G')$. To prove (1.1), therefore, we need only produce a free basis for $\mathbf{Z}[X]^{W'}$ over $\mathbf{Z}[X]^W$. This puts us in the realm of weights, roots and reflection groups, for which we use [2] as a general reference. Let $\Sigma \subseteq X$ be the root system of G relative to T, Σ^+ the set of positive roots and Π the corresponding basis of simple roots relative to some, fixed, ordering. The condition that G be simply connected is:

2.1. The fundamental weights $\{\lambda_a\}$, defined by $(\lambda_a, b^*) = \delta_{ab}$ $(a, b \in \Pi)$ with $b^* = 2b/(b, b)$, form a basis for X.

We generalize our problem slightly by allowing W' to be any reflection subgroup of W. Let Σ' be the corresponding root system, consisting of the roots orthogonal to the reflecting hyperplanes for W', and W'' the subset of W keeping Σ'^+ positive. Finally, for $v \in W''$ let λ_v denote the product in X of those λ_a for which $a \in \Pi$ and $v^{-1}a < 0$, and $e_v = \Sigma x^{-1} v^{-1} \lambda_v \in \mathbf{Z}[X]$, the sum over $x \in W'(v)\backslash W'$ with $W'(v)$ denoting the stabilizer of $v^{-1}\lambda_v$ in W'.

THEOREM 2.2. *Assume G simply connected and the other notations as above. Then $\mathbf{Z}[X]^{W'}$ is free over $\mathbf{Z}[X]^W$ with $\{e_v | v \in W''\}$ as a basis.*

Remarks 2.3. (a) Observe that each $v^{-1}\lambda_v$ is dominant for Σ'. For $(v^{-1}\lambda_v, a) = (\lambda_v, va) \geq 0$ since $va > 0$ for all $a \in \Sigma'^+$. It follows from (2.2) and the above discussion that (1.1) holds with a basis consisting of those irreducible representations of G' for which the highest weights are $\{v^{-1}\lambda_v | v \in W''\}$. It also follows that the rank is $|W|/|W'|$, in (1.1) or in (2.2), either by Galois theory or by (2.5(a)) below. (b) In the principal case in which $W' = \{1\}$, in which G' is a torus in (1.1), we get $\mathbf{Z}[X]$ free over $\mathbf{Z}[X]^W$ with $\{w^{-1}\lambda_w | w \in W\}$ as a basis.

MAIN LEMMA 2.4. *Let $\{e_v\}$ be as above and $\{f_v\}$ ($v \in W''$) any collection of elements of $\mathbf{Z}[X]^{W'}$. Set $D = \det ue_v$, $E = \det uf_v$ ($u, v \in W''$).*

(a) *$D \neq 0$.*

(b) *D divides E and the ratio is in $\mathbf{Z}[X]^W$.*

Granted this lemma, we may prove (2.2) as follows. If $f \in \mathbf{Z}[X]^{W'}$, then the system $\Sigma a_v ue_v = uf$ has a unique solution for $a_v \in \mathbf{Z}[X]^W$, hence the equation $\Sigma a_v e_v = f$ does also, whence (2.2).

It remains to prove (2.4).

LEMMA 2.5. *Let everything be as above.*

(a) *W'' is a system of representatives for W/W'.*

(b) *If Σ' has a basis consisting of a subset of Π, then $\ell(ux) = \ell(u) + \ell(x)$ for $u \in W''$, $x \in W'$.*

Here $\ell(w)$ denotes the number of positive roots made negative by w. Fix $w \in W$. Then $w^{-1}\Sigma^+ \cap \Sigma'$ and Σ'^+ are two positive systems for Σ', hence (*) they are congruent under a unique $x \in W'$. Then $u = wx^{-1} \in W''$ and $w = ux \in W'' \cdot W'$. Conversely, if w has this form, we may work backwards to conclude that x satisfies (*), hence is uniquely determined. This proves (a). The number of roots in Σ'^+ made negative by ux as in (b) is $\ell(x)$ since x fixes Σ' and u fixes the signs of the roots in Σ', while the number in $\Sigma^+ - \Sigma'^+$ is $\ell(u)$ since x fixes this set, whence (b).

LEMMA 2.6. *Assume as before and that $w \in W$ keeps $\Sigma^+ - \Sigma'^+$ positive. Then $w \in W'$, in fact w is in the subgroup generated by the simple reflections that W' contains.*

Assume w as given, $w \neq 1$. Then $wa < 0$ for some simple root a, so that $\ell(ww_a) < \ell(w)$. By our assumption $a \in \Sigma'^+$, so that w_a preserves $\Sigma^+ - \Sigma'^+$ and hence ww_a keeps it positive. By induction on $\ell(w)$ we conclude that ww_a is in the above subgroup, whence w is also.

LEMMA 2.7. *For $v \in W''$ we have $vW(v) \subseteq W''W'(v)$.*

Recall that $W(v)$, for example, denotes the stabilizer of $v^{-1}\lambda_v$ in W. As is known, this is a reflection group. Let $\Sigma(v)$ be the corresponding system of roots, those orthogonal to v. We have $v\Sigma'(v) = v\Sigma' \cap v\Sigma(v)$. Hence $v(\Sigma'^+ - \Sigma'^+(v))$ is disjoint from $(v\Sigma(v))^+$, and it is positive since $v \in W''$. Now if $w \in W(v)$, then $vwv^{-1} \in {}^vW(v)$, the group corresponding to the root system $v\Sigma(v)$, which is the subset of Σ orthogonal to λ_v and hence is like Σ' in (2.5(b)) since λ_v is dominant. By the above disjointness, $vwv^{-1} \cdot v(\Sigma'^+ - \Sigma'^+(v)) > 0$. If we write $vw = ux$ as in (2.5(b)), this yields $x(\Sigma'^+ - \Sigma'^+(v)) > 0$ since u fixes signs on Σ'. Thus $x \in W'(v)$ by (2.6) with Σ, Σ' there replaced by $\Sigma', \Sigma'(v)$ here, whence (2.7).

LEMMA 2.8. *For each root a let n_a denote the number of pairs in W/W' interchanged by left multiplication by w_a.*

(a) *n_a is constant on W-conjugacy classes of roots.*

(b) *If a is simple then n_a is the number of v's in W'' such that $v^{-1}a < 0$.*

If a and b are conjugate, then so are w_a and w_b, hence also their left multiplications on W/W', whence (a). In (b) let w_a fix vW'. Then $v^{-1}w_a v \in W'$, whence $v^{-1}a \in \Sigma'$ and $v^{-1}a > 0$ since $v \in W''$. Now assume w_a does not fix vW', i.e. $v^{-1}a \notin \Sigma'$. Then $v\Sigma'^+$ is disjoint from a and positive, whence $w_a v\Sigma'^+$ is also positive and $w_a v \in W''$. Now just one of $v^{-1}a$, $(w_a v)^{-1}a$ is negative. Thus $v^{-1}a < 0$ for exactly n_a choices of $v \in W''$.

LEMMA 2.9. *If D is as in (2.4) and n_a as in (2.8) then D has $\Pi\lambda_a^{n_a}$ ($a \in \Pi$) as its unique highest term and $\pm\Pi\lambda_a^{-n_a}$ as its unique lowest term.*

This is relative to the usual partial order in which $\lambda > \mu$ denotes that $\lambda\mu^{-1}$ is a product of positive roots. Let A denote the matrix (ue_v). Recall that $ue_v = \Sigma ux^{-1}v^{-1}\lambda_v$, summed over $x \in W'(v)\backslash W'$. Consider the vth column of A. We have $\lambda_v \geq ux^{-1}v^{-1}\lambda_v$ for all terms there. We claim that equality can hold on or above the diagonal only for the term with $u = v$ and $x \in W'(v)$, if we order the rows so that u is above u' whenever $\ell(u) < \ell(u')$. Assume equality. Then $v^{-1}ux^{-1} \in W(v)$ by definition, so that $ux^{-1} \in W''W'(v)$ by (2.7), and $x \in W'(v)$ by (2.5(a)), so that $uv^{-1}\lambda_v = \lambda_v$. Thus uv^{-1}, hence also vu^{-1}, is in the group generated by the reflections for the simple roots orthogonal to λ_v, which are those kept positive by v^{-1} by the definitions. Applying (2.5(b)) to this situation we get $\ell(u^{-1}) = \ell(v^{-1}) + \ell(vu^{-1})$. On or above the diagonal where $\ell(u) \leq \ell(v)$ this can hold only if $\ell(vu^{-1}) = 0$, whence $u = v$ and our claim. It follows that $D = \det A$ has $\Pi\lambda_v$ as its unique highest term. Now λ_a ($a \in \Pi$) makes a contribution to λ_v just when $v^{-1}a<0$. Thus by (2.8(b)) the highest term is as in (2.9). Now each $w \in W$ permutes the rows of A by (2.5(a)) and the invariance of e_v under W', hence fixes D up to sign. It follows that there is a unique lowest term, $\pm w_0\Pi\lambda_a{}^{n_a}$, with w_0 the element of W that makes all positive roots negative. Now if $b = -w_0a$ then $n_a = n_b$ by (2.8(a)). Thus the lowest term is as in (2.9), as required.

Consider now (2.4). We show that $D_1 = \Pi(a^{1/2} - a^{-1/2})$ $(a \in \Sigma^+)$ divides E and that $D_1 = D$. Assume $a \in \Sigma^+$. As noted earlier there are n_a pairs of rows of (uf_v) which are interchanged by w_a. If we subtract row w_au from row u for such a pair then all entries of the result are divisible by $a-1$ since $w_a\lambda = \lambda a^n$, $n = -(\lambda, a^*)$, for $\lambda \in X$. Thus $(a-1)^{n_a}$ divides E, and since $\mathbf{Z}[X]$ is a u.f.d., so do $\Pi(a-1)^{n_a}$ and D_1. In particular D_1 divides D. To prove $D_1 = D$ we need only show that the highest and lowest terms match up, i.e. by (2.9), that $\Sigma n_a a$ $(a>0) = \Sigma n_a\lambda_a$ $(a \in \Pi)$, with the operation of X now written as addition. If s denotes the left side and b a simple root then w_b maps b on $-b$ and permutes the other positive roots. Thus $(1-w_b)s = 2n_bb$, and $(s, b^*) = 2n_b$ by the formula for a reflection, so that s equals the right side by (2.1). Finally, each $w \in W$ acts on the rows of (ue_v) and (uf_v) just as it does on W/W', hence fixes E/D. This proves (2.4), hence also (2.2) and (1.1).

§3. PROOF OF (1.2) AND (1.3)

In this section G is a simply connected group, T is a maximal torus, and the other notations of §2 are used. Further r_a $(a \in \Pi)$ denotes the irreducible representation of G with highest weight λ_a, so that $R(G)$ is a polynomial algebra in the r_a's. If z is in the center of G, then $r_a(z) = \lambda_a(z)$. id. Thus there is a natural action of z on $R(G)$ and $\mathbf{Z}[X]$ with their scalars extended from $\mathbf{Z}$ to $\mathbf{C}$ such that $zr_a = \lambda_a(z)r_a$ and $z\lambda_a = \lambda_a(z)\lambda_a$ for all a. Observe that z fixes roots and commutes with W. We call z a pseudoreflection if it is one on $\Sigma\,\mathbf{C}\,r_a$ or $\Sigma\,\mathbf{C}\,\lambda_a$, i.e. if $\lambda_a(z) = 1$ for every a but one.

MAIN LEMMA 3.1. *Let G be simply connected and Z a subgroup of the center of G. Then the following conditions are equivalent.*

(a) *$R(G')^Z$ is free over $R(G)^Z$ for every connected subgroup G' of maximal rank.*

(b) *$R(T)^Z$ is free over $R(G)^Z$ for some maximal torus T.*

(c) *$R(G)^Z$ is a polynomial algebra over $\mathbf{Z}$.*

(d) *Z is a direct product of the centers of a number of the simple components of G of type* Spin_{2r+1}.

(e) *Z is generated by pseudoreflections.*

(f) *$R(G)^Z$ has a generating set of the form $\{r_a{}^{m_a} \mid a \in \Pi\}$.*

(g) *X^Z has a basis of the form $\{m_a\lambda_a\}$.*

(h) *(X^Z, W, Σ_1) is the data for a simply connected group for some choice of an abstract root system $\Sigma_1 \subseteq X^Z$.*

Consider now (1.2) in which, as noted earlier, G may be assumed semisimple. Since every semisimple group may be written G/Z with G and Z as in (3.1), and since $R(G)^Z, R(T)^Z, \ldots$ have the same significance for G/Z as $R(G), R(T), \ldots$ have for G, Theorem 1.2 follows from the equivalence of (a), (b), (c) and (d) of (3.1).

We first prove the equivalence of the last four parts of (3.1), which have been added mainly for convenience. If (e) holds and Z acts as a product of cyclic groups, the one on r_a being of order m_a, say, then $R(G)^Z = \mathbf{Z}[r_a\text{'s}]^Z = \mathbf{Z}[r_a{}^{m_a}\text{'s}]$, whence (f). Conversely, if this equation holds then Z is a subgroup of the above product, is the whole product in fact since otherwise some nontrivial character $\Pi\lambda_a{}^{d_a}$ $(0 \le d_a < m_a)$ would vanish on Z and $\Pi r_a{}^{d_a}$ would contradict the last equation, whence (e). Since $\Pi r_a{}^{d_a}$ is in $R(G)^Z$ if and only if $\Sigma\, d_a\lambda_a \in X^Z$ (additive notation here), (f) and (g) are equivalent. Observe that in (h) the elements of Σ_1 are multiples of those of Σ since their directions are determined by the reflections of W. If (g) holds then $m_aa = (1-w_a)m_a\lambda_a \in X^Z$ and $(m_aa, (m_bb)^*)$ is always integral since $\{(m_bb)^*\}$ is a basis for the dual of X^Z. It readily follows that $\{m_aa \mid a \in \Pi\}$ is a basis for a root system Σ_1 for which (h) holds. Conversely, if (h) holds and $\{m_aa\}$ is a basis for Σ_1, then $\{m_a\lambda_a\}$ is the corresponding basis of X^Z (see 2.1), whence (g).

Next we prove that (e) $\Rightarrow$ (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (e). If (e) holds, so does (h) and then also (a) by (1.1) which in (2.2) has been reduced to a theorem about (X, W, Σ). Clearly (a) implies (b). Assume (b). Then $A = \mathbf{C}R(T)^Z$ is free, hence also integral, over $B = \mathbf{C}R(G)^Z$. Localize B at the point q of Spec B where all r_a's are 0 and A at a point p of Spec A above q. The first condition makes sense since each r_a has some power in B. We now invoke a result of Auslander–Buchsbaum–Serre.

3.2 A (commutative, Noetherian) local ring R is regular if and only if its cohomological dimension $d(R)$ is finite.

Now let $\{x_1, x_2, \ldots, x_r\}$ with $r = |\Pi|$, the rank of G, be a basis for X^Z, imbedded in A in the natural way, and $x_i(p) = c_i$. Then A_p is the tensor product of r algebras, the ith equal to $\mathbf{C}[x_i, x_i^{-1}]$ localized at $x_i = c_i$, hence is regular, whence $(d(A_p) < \infty$ by (3.2). Since A_p is free over B_q (with basis any basis for $R(T)^Z$ over $R(G)^Z$), $d(B_q) \leq d(A_p)$, so that B_q is regular by (3.2). Thus $\dim_{\mathbf{C}} m/m^2 = r$, if m denotes the maximal ideal of B_q. The monomials $\Pi r_a^{d_a}$ that lie in m form a multiplicative semigroup. Let C be its minimal generating set, consisting of those elements that are not products of others. Clearly B is a basis for m/m^2 over $\mathbf{C}$ so that $|B| = r$. However for each $a \in \Pi$ some $r_a^{m_a}$ lies in B. Thus B consists of the $r_a^{m_a}$'s, and $R(G)^Z$ is a polynomial algebra on the $r_a^{m_a}$'s, whence (f) and also (c). Now assume (c). The free generating set for $R(G)^Z$ may be taken in the ideal m just considered. Then the proof just given shows that (f) holds, hence also (e).

It remains only to prove the equivalence of (d) and (e). For this we may assume that G is simple since if $z \in Z$ is a pseudoreflection it acts nontrivially on just one r_a, hence belongs to some simple component of G. Let V be the universal covering space for T, a real Euclidean space, and for convenience take the character values $\lambda(v)$ to be in $\mathbf{R}/\mathbf{Z}$ rather than in the complex numbers of norm 1. Then there is the famous fundamental simplex $S: \{v \in V \mid a(v) \geq 0\ (a \in \Pi),\ h(v) \leq 1\}$. Here $h = \Sigma\, h_a a$ is the highest root. This sum is to be taken over Π and similarly for the sums on $a, b, \ldots$ that follow. The center of G is represented in S by 0 and the vertices z_a of S corresponding to a's for which $h_a = 1$. For any such we have

$$b(z_a) = \delta_{ba} \qquad (b \in \Pi). \tag{3.3}$$

For a again arbitrary write

$$\lambda_a = \Sigma\, n_{ab} b \qquad (n_{ab} \in \mathbf{Q}). \tag{3.4}$$

We claim that for the dual root system Σ^*, in which a is replaced by $2a/(a, a)$ and similarly for λ_a, the corresponding equation reads

$$\lambda_a^* = \Sigma\, n_{ba} b^*. \tag{3.5}$$

For substituting the definitions into (3.4) we get (3.5) with the coefficient of b^* equal to $n_{ab}(b, b)/(a, a)$. But $(\lambda_a, \lambda_c) = n_{ac}(c, c)/2$ by (3.4) and (2.1), whence $n_{ac}(c, c) = n_{ca}(a, a)$ by symmetry. The coefficient of b^* thus becomes n_{ba}, whence (3.5). Now assume that $z_a \in Z$ acts as a pseudoreflection on $\mathbf{C}R(G) = \mathbf{C}[r_a\text{'s}]$. Then $\lambda_c(z_a)$ is integral with just one exception, say for $c = b$. But $\lambda_c(z_a) = n_{ca}$ by (3.4). Thus (by (3.5)) $n_{ba} b^*$, hence also some submultiple of b^*, is a weight. This implies that Σ^* is of type C_r and b^* is the unique long simple root, as is well known and proved thus: in any other case there is a simple root c^* such that $(b^*, c^{**}) = -1$, so that b^* is primitive as a weight. Then Σ is of type B_r (and $G = \mathrm{Spin}_{2r+1}$), and a is the long root at the end of the Dynkin diagram and $\{1, z_a\}$ is the center of G since this a is the only simple root for which $h_a = 1$. Conversely, if Σ and a are as just mentioned it can be verified that λ_a^* in (3.5) has exactly one nonintegral coefficient so that z_a is a pseudoreflection. Thus (d) and (e) are equivalent, and (3.1) is completely proved.

Remarks 3.6. (a) For a proof of the equivalence of (b), (c) and (e) in a more general setting see [4], from which our proof that (b) implies (c) is taken. We could avoid the other heavy commutative algebra used there because of the simple action of Z in our case. (b) The geometric essence of the equivalence of (c) and (e) in its general form is that, in an algebraic or analytic variety acted on by a finite group Z of order not divisible by the characteristic a nonsingular point p remains nonsingular in the quotient space if and only if Z^p acting on the tangent space at p is generated by pseudoreflections.

Finally, we consider (1.3). Since every irreducible representation of G is trivial on the unipotent radical of G, we may assume G reductive. Then we may reduce (1.3) to properties of abstract root systems and reflection groups, as we reduce (1.1) to (2.2), properties which have been proved above.

REFERENCES

1. J. F. ADAMS: *Lectures on Lie Groups*. Benjamin, New York (1969).
2. N. BOURBAKI: *Groupes et algèbres de Lie*, Chapters IV, V and VI. Hermann, Paris (1968).
3. H. V. PITTIE: Homogeneous vector bundles on homogeneous spaces, *Topology* **11** (1972) 199–203.
4. J.-P. SERRE: *Colloque d'algèbre*, no. 8. Ecole Normale Supérieure de Jeunes Filles, Paris (1967).

University of California, Los Angeles